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Noncommutative/Nonlinear BPS Equations without Zero Slope Limit

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Abstract

It is widely believed that the linearly realized BPS equation in the non-commutative space is related to the non-linearly realized BPS equation in the commutative space in the zero slope limit. We shall show that the relation also holds without taking the zero slope limit as is expected from arguments of the BPS equation for the non-Abelian Born-Infeld theory. This is regarded as an evidence for the relation between the two BPS equations.

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1 Introduction and summary

Recently the string theory turns out to be fertile. It contains many important concepts in physics. Among other things D-brane in the string theory with the background NS-NS 2-form B_{ij} has two effective theories: the ordinary Born-Infeld theory when the Pauli-Villars regularization is adopted [1] and the non-commutative Born-Infeld theory when the point-splitting regularization is adopted [2, 3]. Since the method of regularizations should not change the physical S-matrices, it was discussed in [4] that these two descriptions should be related by field redefinitions (called the Seiberg-Witten map).

This relation is also explored from the classical solutions. In fact, many solitons and instantons were constructed in the non-commutative space [5, 6, 7, 8, 9, 10, 11] for many interesting properties by themselves [12, 13, 14, 15, 16] and their relations to the commutative space [4, 17, 18, 19, 20]. In the most cases except [17, 20] the relation was discussed in the zero slope limit. In fact, since the linearly realized BPS equation is directly obtained from the Yang-Mills theory which is the zero slope limit of the Born-Infeld theory, it is widely believed that the linearly realized BPS equation in the non-commutative space is related to the non-linearly realized BPS equation in the commutative space in the zero slope limit by the Seiberg-Witten map. This was implicitly pointed out in [4] by rewriting the non-linear BPS equation in the zero slope limit in terms of the open string moduli: the open string metric G_{ij} and the non-commutativity parameter θ^{ij}, which originally appears in the non-commutative BPS equation. However, it is still difficult to show the relation explicitly because we do not have enough freedom in the description to connect these two BPS equations.

On the other hand, it was shown in [21, 22] that although the linear BPS equation is obtained directly from the Yang-Mills theory, it also reproduces the equation of motion of the Born-Infeld theory if we adopt the symmetrized prescription [23]. Though this was shown in the non-Abelian case, we can extend it to the non-commutative case straightforwardly. The fact that the linear BPS equation in the non-commutative space is unchanged in the zero slope limit implies that taking the zero slope limit is unnecessary also in their commutative counterpart because the Seiberg-Witten map is defined independently of the slope α'.

In this paper we shall rewrite the non-linear BPS equation in terms of the open string moduli without taking the zero slope limit. We shall show that even we do not take the zero slope limit the non-linear BPS equation in the open string moduli gives completely the same form as what obtained in the zero slope limit.

This result shows that the zero slope limit is unnecessary in discussing the relation. More importantly, this result gives us a strong evidence for the conjecture that the linear BPS equation in the non-commutative space is related to the non-linear BPS equation in the commutative space, because the invariant nature of the linear BPS equation in the non-commutative space under the zero slope limit is perfectly reproduced in the commutative side.

In the next section, we shall explicitly rewrite the non-linear BPS equation in terms of the open string moduli without taking the zero slope limit. And we shall discuss the physical implications and their applications in the final section.

2 Nonlinear BPS equation in the open string moduli

In this section we shall rewrite the non-linear BPS equation in terms of the open string moduli. First let us recall the linear supersymmetries and non-linear supersymmetries of gauginos [24, 25, 23].

$$
\begin{aligned}
\delta_{\mathrm{L}}\lambda_+ &= \frac{1}{2\pi\alpha'}M^+_{ij}\sigma^{ij}\eta, &(1)\\
\delta_{\mathrm{L}}\lambda_- &= \frac{1}{2\pi\alpha'}M^-_{ij}\bar\sigma^{ij}\bar\eta, &(2)\\
\delta_{\mathrm{NL}}\lambda_+ &= \frac{1}{4\pi\alpha'}\Big(1-\mathrm{Pf}\,M+\sqrt{1-\mathrm{Tr}\,M^2/2+(\mathrm{Pf}\,M)^2}\Big)\eta^*, &(3)\\
\delta_{\mathrm{NL}}\lambda_- &= \frac{1}{4\pi\alpha'}\Big(1+\mathrm{Pf}\,M+\sqrt{1-\mathrm{Tr}\,M^2/2+(\mathrm{Pf}\,M)^2}\Big)\bar\eta^*, &(4)
\end{aligned}
$$

where M denotes

$$
M = 2\pi\alpha'(F+B) \tag{5}
$$

with the field strength F_{ij} and the background NS-NS 2-form B_{ij}. Hereafter we shall set $2\pi\alpha'=1$ for simplicity, however we can restore it on the dimensional ground. At the infinity the field strength vanishes and the combination of the unbroken supersymmetries is given as

$$
B^+_{ij}\sigma^{ij}\eta+\frac{1}{2}\Big(1-\mathrm{Pf}\,B+\sqrt{1-\mathrm{Tr}\,B^2/2+(\mathrm{Pf}\,B)^2}\Big)\eta^*=0. \tag{6}
$$

The non-linear BPS equation is the condition of preserving these supersymmetries:

$$
\frac{M^+}{1-\mathrm{Pf}\,M+\sqrt{1-\mathrm{Tr}\,M^2/2+(\mathrm{Pf}\,M)^2}}=\frac{B^+}{1-\mathrm{Pf}\,B+\sqrt{1-\mathrm{Tr}\,B^2/2+(\mathrm{Pf}\,B)^2}}. \tag{7}
$$

We shall rewrite this non-linear BPS equation in terms of the open string moduli. However before doing it we shall first rewrite the non-linear BPS equation (7) into a simpler form. First note from eq. (7), the matrix M^+ must be proportional to B^+:

$$
M^+ = fB^+. \tag{8}
$$

Rewriting eq. (7) into a scalar equation as

$$f\Big(1-\mathrm{Pf}\,B+\sqrt{1-\mathrm{Tr}\,B^2/2+(\mathrm{Pf}\,B)^2}\Big)-(1-\mathrm{Pf}\,M)=\sqrt{1-\mathrm{Tr}\,M^2/2+(\mathrm{Pf}\,M)^2}, \tag{9}$$

and taking the square of it, eq. (7) is reduced to a much simpler form [26, 17]:

$$\frac{M^+}{1-\mathrm{Pf}\,M}=\frac{B^+}{1-\mathrm{Pf}\,B}. \tag{10}$$

Here we have used the following identities

$$\mathrm{Tr}\,M^2 = \mathrm{Tr}(M^+)^2+\mathrm{Tr}(M^-)^2, \tag{11}$$
$$4\,\mathrm{Pf}\,M = -\mathrm{Tr}(M^+)^2+\mathrm{Tr}(M^-)^2. \tag{12}$$

Note that if we further use the identity,

$$\mathrm{Pf}(F+B)=\mathrm{Pf}\,F+\mathrm{Pf}\,B-\mathrm{Tr}\,F\tilde{B}/2, \tag{13}$$

eq. (10) now reads

$$F^+(1-\mathrm{Pf}\,B)=B^+(\mathrm{Tr}\,F\tilde{B}/2-\mathrm{Pf}\,F). \tag{14}$$

Hereafter we shall rewrite eq. (14) in terms of the open string moduli: the open string metric G_{ij} and the non-commutativity parameter θ^{ij}. The open string moduli is related to the closed string moduli as [4]

$$\frac{1}{G}+\theta=\frac{1}{g+B}. \tag{15}$$

Since we adopt the flat metric for the closed string metric $g_{ij}=\delta_{ij}$, we can express the open string metric G_{ij} and the non-commutativity parameter θ^{ij} in terms of the B-field:

$$G_{ij} = \delta_{ij}-(B^2)_{ij}, \tag{16}$$
$$\theta^{ij} = \frac{-B_{ij}-\tilde{B}_{ij}\,\mathrm{Pf}\,B}{\det(1+B)}. \tag{17}$$

Since in eq. (14) the self-dual projection appears, we also expect it to appear in the BPS equation in the open string moduli. As we know from [4] the easiest way to write down the self-dual projection is neither in the covariant frame nor in the contravariant frame but in the local Lorentz frame. Hence we have to calculate

$$\underline{F}^+ = (E^tFE)^+, \tag{18}$$
$$\underline{\theta}^+ = \left(\frac{1}{E}\theta\frac{1}{E^t}\right)^+. \tag{19}$$

Here the vierbein is defined as $EGE^t = 1$. From the metric (16) we find that the vierbein is given as

$$E = (1+B)^{-1}. \tag{20}$$

In calculating the self-dual projection of the field strength, we shall go to a special frame where B has the canonical form as in [4]:

$$B = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & -b_2 & 0 \end{pmatrix}. \tag{21}$$

In this frame $\underline{F}^+$ is given as

$$\underline{F}^+ = \left(\frac{1}{1-B}F\frac{1}{1+B}\right)^+ = \frac{1}{(1+b_1^2)(1+b_2^2)} \begin{pmatrix} 0 & f_1 & f_2 & f_3 \\ -f_1 & 0 & f_3 & -f_2 \\ -f_2 & -f_3 & 0 & f_1 \\ -f_3 & f_2 & -f_1 & 0 \end{pmatrix}, \tag{22}$$

where f_1, f_2 and f_3 denote

$$\begin{aligned} 2f_1 &= (1+b_2^2)F_{12} + (1+b_1^2)F_{34}, & (23)\\ 2f_2 &= (1-b_1b_2)(F_{13}-F_{24}) + (b_1+b_2)(F_{14}+F_{23}), & (24)\\ 2f_3 &= (1-b_1b_2)(F_{14}+F_{23}) - (b_1+b_2)(F_{13}+F_{24}). & (25) \end{aligned}$$

We easily identify terms proportional to F^+ and B^+, however there are still other terms to be identified as $B^+F^+ - F^+B^+$:

$$\left(\frac{1}{1-B}F\frac{1}{1+B}\right)^+ = \frac{(1-\mathrm{Pf}\,B)F^+ - (\mathrm{Tr}\,F\tilde{B})B^+/2 + (B^+F^+ - F^+B^+)}{\det(1+B)}. \tag{26}$$

On the other hand, $\underline{\theta}^+$ is much easier to calculate. We find

$$\underline{\theta}^+ = \Big((1+B)\theta(1-B)\Big)^+ = -B^+, \tag{27}$$

where we have used

$$\begin{aligned} B^3 &= (\mathrm{Tr}\,B^2)B/2 + (\mathrm{Pf}\,B)\tilde{B}, & (28)\\ B\tilde{B}B &= -(\mathrm{Pf}\,B)B, & (29) \end{aligned}$$

in the calculation. Since we are considering the non-linear BPS equation (7) which implies that F^+ is proportional to B^+ (8), it is possible to add $(B^+F^+ - F^+B^+)$ to the eq. (14) freely because $B^+F^+ - F^+B^+ = 0$. Collecting all our results and comparing with eq. (14) we find it remains to rewrite $\mathrm{Pf}\, F$ into the local Lorentz frame:

$$\mathrm{Pf}\,\underline{F} = \frac{\mathrm{Pf}\,F}{\det(1+B)}. \tag{30}$$

Therefore finally our non-linear BPS equation (7) is rewritten as

$$\underline{F}^+ = \underline{\theta}^+\,\mathrm{Pf}\,\underline{F}, \tag{31}$$

which amazingly is the same form as what obtained in the zero slope limit [4].

3 Physical implications and further directions

First, our analysis in this paper is important in the conceptual sense. We give another strong evidence for the fact that the linear BPS equation in the non-commutative space is mapped to the non-linear BPS equation in the commutative space, because the invariant nature of the linear BPS equation in the non-commutative space under the zero slope limit is perfectly reproduced in the commutative side.

Secondly, so far we have neglected the corrections to the Born-Infeld theory. The solutions of the linear BPS equation are also solutions for the Born-Infeld theory with corrections [27]. However we do not know a similar argument for the non-linear BPS equation. To clarify this point is an interesting direction.

Thirdly, as a technical application this rewriting enables us to find an instanton solution to the non-linear BPS equation for a general B-field background. In [17] the solution is constructed under the condition $B^- = 0$, because otherwise the solution for a general B-field background is very intricate. However the non-linear BPS equation is now rewritten as (31), which is completely solved in [4]. Hence it is possible to read off the solution for the non-linear BPS equation directly:

$$A_i = \theta^+_{ij} x^j \cdot \frac{1}{4}\left(-1 \pm \sqrt{1 + \frac{32C}{R^4}}\right), \tag{32}$$

with $2\theta^+_{ij} = -(1 - \mathrm{Tr}\,B^2/2 + \mathrm{Pf}\,B)B_{ij} - (1 - \mathrm{Pf}\,B)\tilde{B}_{ij}$ and $R^2 = x^i(1-B^2)_{ij}x^j$. Though the solution should be independent of θ^- as noted in [4], this does not mean it is possible to neglect B^- as well. In fact our solution (32) has a non-trivial dependence on B^-. In this way the commutative counterparts of the non-commutative Abelian instanton and monopole [5, 11] with or without taking the zero slope limit are all constructed [4, 20].

Finally, since we have rewritten the non-linear BPS equation completely in terms of the open string moduli, we should expect the tension of the exact monopole solution found in [20] also has a trivial dependence on α'. However, it was discussed in [11] that this is not the case and there might be a discrepancy between the non-commutative and commutative viewpoints. Therefore we shall reconsider it here.

Since the constant Higgs field is not changed in the Seiberg-Witten map, what to compare with the tension of the non-commutative monopole should be the tension per the unit length of the Higgs field of the non-linear BPS monopole. This leads [11] to consider the quantity $T_{\mathrm{D1}}/\sin\phi$. In [20] we consider the target space rotation only when the target space is flat with the rotation angle $\tan\phi = (2\pi\alpha')B$ and the non-commutativity parameter $\theta = (2\pi\alpha')^2 B/(1+(2\pi\alpha')^2B^2)$. Note that even in the open string moduli, the open string metric parallel to B remains flat, which makes it possible to rotate in this direction without any factors. In this case we find the tension to be compared is

$$\frac{T_{\mathrm{D1}}}{\sin\phi} = \frac{1}{2\pi\alpha' g_s}\frac{\sqrt{1+(2\pi\alpha')^2B^2}}{2\pi\alpha' B} = \frac{1}{G_s\theta}, \tag{33}$$

which seems consistent with what found in the non-commutative space [11] except a numerical factor probably due to the difference in notation. Here the final result (33) does not depend on α' as we expected.

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